Functions 4024
Compiled by : Mustafa Asif

26. Function notation	• use function notation, e.g. $f(x) = 3x - 5$, $f: x \mapsto 3x - 5$, to describe simple functions • find inverse functions $f^{-1}(x)$	

Video lectures for understanding

https://www.youtube.com/watch?v=u1YQVzrgYDg

https://www.youtube.com/watch?v=zpF9nbjResY

https://www.youtube.com/watch?v=kvGsIo1TmsM

Unit-9

Functions

Key Points

What is a Function?

A function relates an input to an output. It is like a machine that has an input and an output and the output is related somehow to the input.

f(x)	f(x) is the classic way of writing a function. And there are other ways, as you will see!

Input **Relationship, Output**

We will see many ways to think about functions, but there are always three main parts:

- The input
- The relationship
- The output

Example: "Multiply by 2" is a very simple function.

Examples of Functions: x^2 (squaring) is a function, x^3+1 is also a function

Here are the three parts: Input	*Relationship*	Output
0	× 2	0
1	× 2	2
7	× 2	14
10	× 2	20
..	..	..

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Names of Function

First, it is useful to give a function a name. The most common name is "f", but we can have other names like "g" ... or even "marmalade" if we want. But let's use "f"

Example: $f(x) = x^2$:

- an input of 4
- becomes an output of 16.

In fact we can write $f(4) = 16$. The "x" is just a Place-Holder!

Don't get too concerned about "x", it is just there to show us where the input goes and what happens to it. It could be anything!

Let function $f(x) = 1 - x + x^2$

Is the same function as:

- $f(q) = 1 - q + q^2$
- $h(A) = 1 - A + A^2$
- $w(\theta) = 1 - \theta + \theta^2$

The variable (x, q, A, etc) is just there so we know where to put the values: $f(2) = 1 - 2 + 2^2 = 3$

Sometimes there is No Function Name

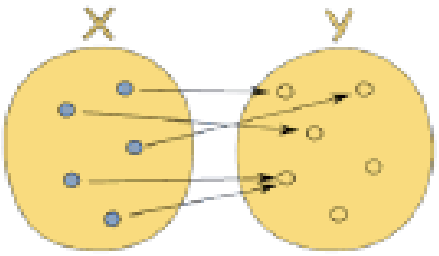
Sometimes a function has no name, and we see something like: $y = x^2$

But there is still: an input (x), a relationship (squaring), and an output (y)

Function has special rules:

- It must work for **every** possible input value
- And it has only **one relationship** for each input value

This can be said in one definition:



Formal Definition of a Function

A function relates **each element** of a set with **exactly one** element of another set (possibly the same set).

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The Two Important Things!

1. "...each element..." means that every element in X is related to some element in Y.

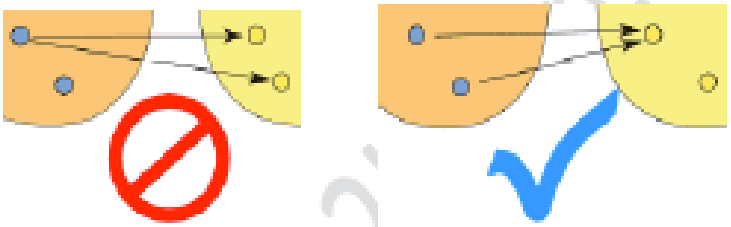
   We say that the function *covers* X (relates every element of it).

   (But some elements of Y might not be related to at all, which is fine.)

2. "...exactly one..." means that a function is *single valued*. It will not give back 2 or more results for the same input.

   So "f(2) = 7 or 9" is not right!

Note: "One-to-many" is not allowed, but "many-to-one" is allowed:



(one-to-many)
This is NOT OK in a function

(many-to-one)
But this is OK in a function

When a relationship does not follow those two rules then it is not a function ... it is still a relationship, just not a function.

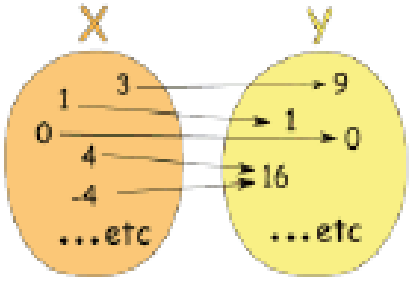
Example: The relationship $x \to x^2$

It is a function, because:

- Every element in X is related to Y
- No element in X has two or more relationships

So it follows the rules.

(Notice how both 4 and -4 relate to 16, which is allowed.)

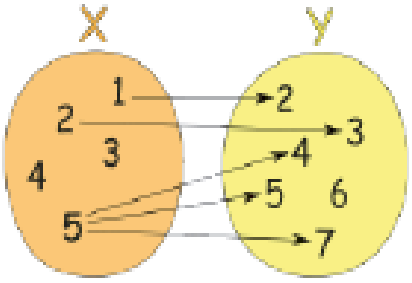


Example: This relationship is not a function:

It is a relationship, but it is not a function, for these reasons:

- Value "3" in X has no relation in Y
- Value "4" in X has no relation in Y
- Value "5" is related to more than one value in Y

(But the fact that "6" in Y has no relationship does not matter)



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Domain, Co-domain and Range

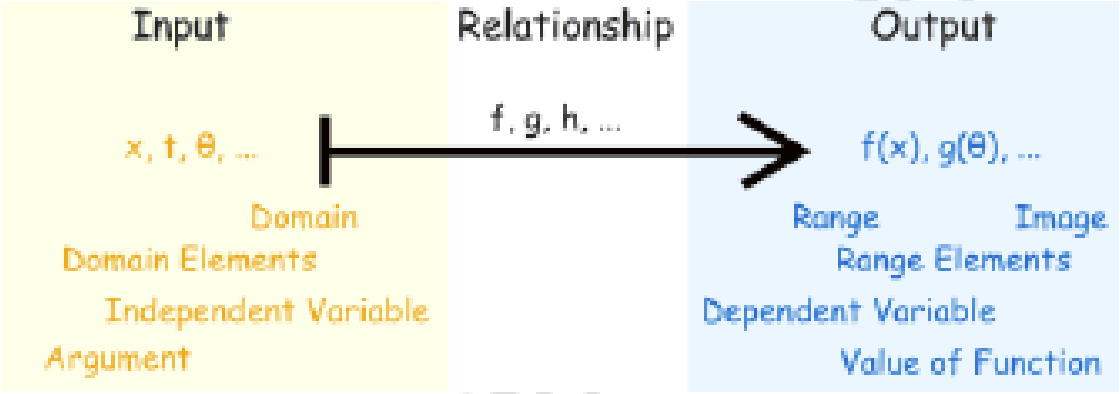
In above examples

- the set "X" is called the **Domain**,
- the set "Y" is called the **Co-domain**, and
- the set of elements that get pointed to in Y (the actual values produced by the function) is called the **Range**.

So Many Names!

Functions have been used in mathematics for a very long time, and lots of different names and ways of writing functions have come about.

Here are some common terms you should get familiar with:



Example: with $z = 2u^3$:

- "u" could be called the "independent variable"
- "z" could be called the "dependent variable" (it depends on the value of u)

Example: with $f(4) = 16$:

- "4" could be called the "argument"
- "16" could be called the "value of the function"

Composite Functions: (For IGCSE)

For two functions f and g, we the composite functions are

i) fg where $fg(x) = f(g(x))$

ii) gf where $gf(x) = g(f(x))$

Inverse of a Composite Function:

$(gf)^{-1} = f^{-1}g^{-1}$ holds for any two one-one functions

1 $f(x) = 5x - 7$ $g(x) = \frac{x+4}{3}$

M/J19/21/7

(a) Find f(6).

..... [1]

(b) Find $g^{-1}(x)$.

$g^{-1}(x) =$ ..... [2]

(c) Given that $f(p) = g(p) - 2$, find p.

$p =$ ..... [3]

(d) $g(5x - 7) = ax + b$.

Find a and b.

$a =$ ..... $b =$ ..... [3]

2 $f(x) = 3(x-2)$ $g(x) = \frac{7x}{4} - 1$

M/J19/22/11

(a) Find g(2).

..... [1]

(b) Find $g^{-1}(x)$.

$g^{-1}(x) =$ ..... [2]

(c) Given that $f(t) = 6$, find t.

$t =$ ..... [2]

(d) $f\left(\frac{7x}{4} - 1\right) = px + q$

Find p and q.

$p =$ ..... $q =$ ..... [3]

3 $f(x) = \frac{7-3x}{2x}$

SP18/01/14

(a) Find $f(4)$.

Answer ..... [1]

(b) Find $f^{-1}(x)$.

Answer $f^{-1}(x) =$ ..... [2]

4 $f(x) = \frac{3}{x+4}$

(a) Find $f(-6)$.

O/N18/11/6

Answer ..... [1]

(b) Find $f^{-1}(x)$.

Answer $f^{-1}(x) =$ ..... [2]

5 $f(x) = \frac{2x+5}{3x}$

(a) Find $f(-2)$. O/N18/12/6

Answer ..... [1]

(b) Find $f^{-1}(x)$.

Answer $f^{-1}(x) =$ ..... [3]

6 $f(x) = 3 - 2x$ $g(x) = 4x^3 - 1$ M/J18/11/15

(a) Find $f(5)$.

Answer ..... [1]

(b) Find $g(-2)$.

Answer ..... [1]

(c) Find and simplify $f(4x^3 - 1)$.

Answer ..... [1]

7 $f(x) = \frac{1}{3x+2}$

(a) Find f(–2). M/J18/12/11

Answer ..... [1]

(b) Find $f^{-1}(x)$.

Answer $f^{-1}(x) =$ ..... [2]

8 $f(x) = 3x + 7$ O/N17/11/7

(a) Find f(3.2).

Answer ..... [1]

(b) Find $f^{-1}(x)$.

Answer $f^{-1}(x) =$ ..... [1]

9 $f(x) = \frac{x}{4}$

(a) Find $f\left(\frac{1}{2}\right)$. O/N17/12/4

Answer ..... [1]

(b) Find $f^{-1}(x)$.

Answer $f^{-1}(x) =$ ..... [1]

10 **(b)** $f(x) = x - 3$ $g(x) = x^2 + 1$ M/J17/11/24(b)

(i) Find $f(-5)$.

Answer ..... [1]

(ii) Find m given that $g(m-3) = 17$.

Answer $m =$ ..... or ..... [3]

11 $f(x) = \frac{3x-k}{4}$ M/J17/12/14

(a) Given that $f(11) = 7$, find the value of k.

Answer $k =$ ..... [2]

(b) Find $f^{-1}(x)$.

Answer $f^{-1}(x) =$ ..... [2]

12 $f(x) = 1-2x$ $g(x) = x+4$ $h(x) = x^2+1$

(a) Find $f(-1)$.

..... [1]

(b) Solve the equation.
$2f(x) = g(x)$

$x =$ ..... [2]

(c) Find $fg(x)$.
Give your answer in its simplest form.

..... [2]

(d) Find $hh(2)$.

..... [2]

(e) Find $f^{-1}(x)$.

$f^{-1}(x) =$ ..... [2]

(f) $\text{hgf}(x) = 4x^2 + px + q$

Find the value of p and the value of q.

$p =$ .....

$q =$ ..... [4]

13 $\text{f}(x) = \frac{3-x}{10}$ O/N16/11/11

(a) Evaluate $\text{f}(-\frac{1}{2})$.

Answer ..... [1]

(b) Find $\text{f}^{-1}(x)$.

Answer $\text{f}^{-1}(x) =$ ..... [2]

14 $f(x) = 4 + 3x$ O/N16/12/10

(a) Find $f\left(-2\frac{1}{2}\right)$.

Answer ..... [1]

(b) Find $f^{-1}(5)$.

Answer ..... [2]

15 A function f is defined by $f: x \mapsto \frac{x+5}{3}$. M/J04/01/11

(a) Given that $f: 1 \mapsto k$, find the value of k.

(b) Given also that $f^{-1}: x \mapsto cx + d$, find the value of c and the value of d.

Answer (*a*) $k =$ ..... [1]

(*b*) $c =$ ..... $d =$ ..... [2]

16 $f(x) = 2x - 9$ M/J16/11/11

(a) Find $f\left(-\frac{3}{4}\right)$.

Answer ..... [1]

(b) Find $f^{-1}(3)$.

Answer ..... [2]

17 (a) The table shows the values of the function $f(x)$ for some values of x.

x	1	2	3	4	5
$f(x)$	5	7	9	11	13

Express the function $f(x)$ in terms of x. M/J16/12/21

Answer $f(x) =$ ..... [1]

(b) $g(x) = \frac{8-3x}{2}$

(i) Evaluate $g(-2)$.

Answer ..... [1]

(ii) Find $g^{-1}(x)$.

Answer $g^{-1}(x) =$ ..... [2]

18 $f(x) = 1 + 4x$ O/N15/12/4

(a) Find $f\left(-\frac{2}{5}\right)$.

Answer ..... [1]

(b) Find $f^{-1}(x)$.

Answer $f^{-1}(x) =$ ..... [1]

19 $f(x) = x^3$ M/J15/21/9

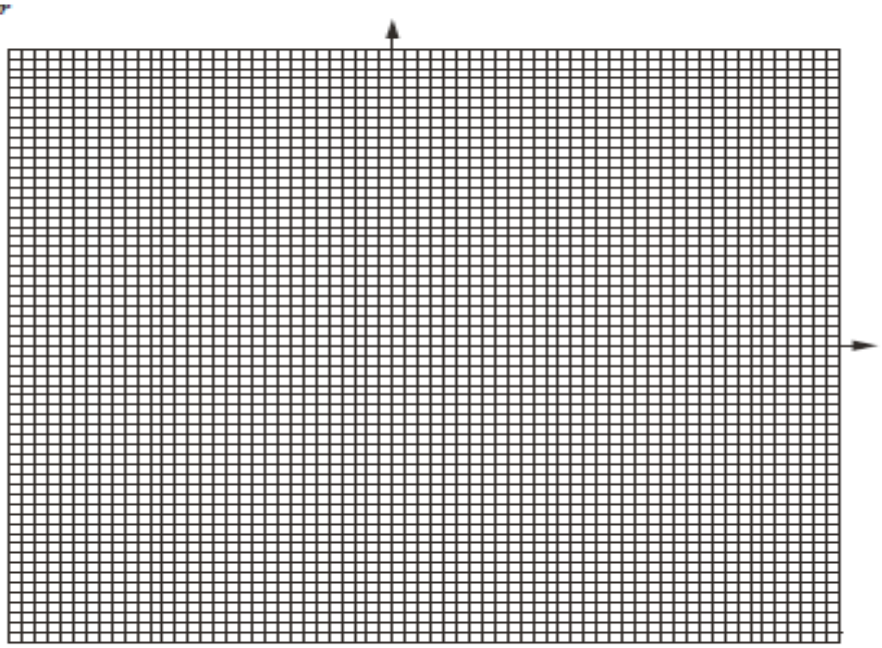
(a) Complete the following table.

x	–3	–2	–1	0	1	2	3
$f(x)$							

[1]

(b) Using a scale of 2 cm to represent 1 unit, draw a horizontal x-axis for $-3 \leqslant x \leqslant 3$.
Using a scale of 2 cm to represent 10 units, draw a vertical y-axis for $-30 \leqslant y \leqslant 30$.
Using your axes, plot the points in the table and join them with a smooth curve.

Answer



[2]

(c) **(i)** Use your graph to solve $f(x) = -15$.

Answer ..... [1]

(ii) Use your graph to find a such that $f^{-1}(a) = 1.7$.

Answer ..... [1]

(iii) Given that $f^{-1}(t) = u$, express t in terms of u.

Answer $t =$ ..... [1]

(iv) By drawing a tangent to $y = f(x)$, estimate the gradient of the curve when $x = 2$.

Answer ..... [2]

(d) (i) Using the same axes draw the line that represents the function $g(x) = 5x + 3$.

[2]

(ii) Hence find the three solutions of the equation $f(x) = g(x)$.

Answer $x =$ ..... or ..... or ..... [2]

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Marking Scheme

1(a)	23	**1**	
1(b)	$3x - 4$ final answer	**2**	**B1** for $3y = x + 4$ **or** $y - \frac{4}{3} = \frac{x}{3}$ **or** $3y - 4$ **or** $x = \frac{y+4}{3}$
1(c)	$\frac{19}{14}$ or $1\frac{5}{14}$ or 1.36 or 1.357 to 1.358	**3**	**B2** for answer $14p = k$ or $kp = 19$ **or** **M1** for $5p - 7 = \frac{p+4}{3} - 2$
1(d)	$a = \frac{5}{3}$ oe $b = -1$	**3**	**B2** for either correct **or** **M1** for $\frac{5x - 7 + 4}{3}$

2(a)	2.5 oe	**1**	
2(b)	$\frac{4x+4}{7}$ oe	**2**	**B1** for $\frac{7x}{4} = y + 1$ or $7x - 4 = 4y$ or $\frac{x}{4} - \frac{1}{7} = \frac{y}{7}$ or $x = \frac{7y}{4} - 1$ or better
2(c)	4	**2**	**B1** for $3(t - 2) = 6$ oe
2(d)	$p = \frac{21}{4}$ oe $q = -9$	**3**	**B2** for $p = \frac{21}{4}$ oe or $q = -9$ or **B1** for $3\left(\frac{7x}{4} - 1 - 2\right)$ oe

3(a)	$-\frac{5}{8}$, or -0.625, **cao**	**1**	
3(b)	$\frac{7}{2x+3}$ **oe**	**2**	**B1** for $2x'y' + 3x = 7$ **oe** (condone swaps of x and 'y') with both variables on the same side.

4(a)	-1.5 oe	**1**	
4(b)	$\frac{3-4x}{x}$ oe final answer	**2**	**B1** for $y(x + 4) = 3$; or $x = \frac{3}{y+4}$; or better

5(a)	$-\frac{1}{6}$ oe	**1**	
5(b)	$\frac{5}{3x-2}$ oe final answer	**3**	**B2** for $3yx - 2y = 5$; or for $x(3y - 2) = 5$ or **B1** for $3xy = 2x + 5$; or for $x = \frac{2y+5}{3y}$; or for $3xy - 2x = 5$; or for $3xy = 2y + 5$

6(a)	-7	**1**	
6(b)	-33	**1**	
6(c)	$5 - 8x^3$ Final answer	**1**	

7(a)	$-\frac{1}{4}$ oe	**1**	
7(b)	$\frac{1-2x}{3x}$ oe final answer	**2**	**M1** for correct first step: $y(3x+2)=1$ or $x=\frac{1}{3y+2}$ or $3x+2=\frac{1}{y}$ or better

8(a)	16.6	**1**	
8(b)	$\frac{x-7}{3}$ oe	**1**	

9	$\frac{1}{2}$ oe nfww	**2**	**B1** for "k" $= \frac{30}{6}$ oe if $y = \frac{k}{x}$ used or FT **M1** for $y =$ (*their k*) / 10 when $y =$ "k" / x used or **M1** for $\frac{1}{6} \times 30 = y \times 10$
9(a)	$\frac{1}{8}$; or 0.125	**1**	

10(b)(i)	-8	1	
10(b)(ii)	-1 or 7 with correct working	3	M1 $(m-3)^2$ correctly expanded to $m^2 - 6m$ +9 or $(m-3)^2 + 1 = 17$ **and** M1 for $m^2 - 6m - 7 = 0$ or $(m-3) = \pm 4$ or SC1 for $m = -1$ or 7 with no working

11(a)	5	**2**	**M1** for $7=\frac{3\times11-k}{4}$ soi
11(b)	$\frac{4x+k}{3}$ or $\frac{4x+5}{3}$ oe final answer	**2**	**FT** *their k* **M1** for correct first step e.g. $x=\frac{3y-k}{4}$ or $4y=3x-k$ or better

IGCSE QUESTION:

12(b)	$-\frac{2}{5}$ oe	**2**	**M1** for $2(1-2x)=x+4$
12(c)	$-2x-7$ final answer	**2**	**M1** for $1-2(x+4)$
12(d)	26	**2**	**B1** for h(5) soi or **M1** for $(x^2+1)^2+1$
12(e)	$\frac{1-x}{2}$ oe final answer	**2**	**M1** for $x=1-2y$ or $2x=1-y$ or $\frac{y}{2}=\frac{1}{2}-x$ or $y-1=-2x$
12(f)	$[p=]-20$ $[q=]26$	**4**	**B3** for $[\text{hgf}(x)]=4x^2-20x+26$ seen and not spoilt by further working or **M1** for $(1-2x)+4$ **M1 dep** for $(\textit{their}\ (5-2x))^2+1$ **B1FT dep** for $25-10x-10x+4x^2$

13	**(a)**	0.35 oe	1	
	(b)	$3-10x$ oe	2 *	**C1** for $10x-3$ or **B1** for 10 "y" = 3 – "x"

14	**(a)**	–3.5 or any equivalent	1	
	(b)	$\frac{1}{3}$	2*	**M1** for $5=4+3x$ or **B1** for $(f^{-1}(x)=)\ \frac{x-4}{3}$ oe or **B1** for $x=\frac{1}{3}$, followed by further work

15(a)	2	1				
(b)	(c =) 3 (x) (d =) - 5	1 1	One correct **or** (f^{-1} : x) $3x$ - 5 seen in working	C1 M1	3	

16	**(a)**	$-10\frac{1}{2}$ oe	1	
	(b)	6	2 *	**B1** for $3 = 2$ 'x' $- 9$ or for $\frac{x+9}{2}$ or $\frac{y+9}{2}$

17	**(a)**	$2x + 3$ oe	1	
	(b) (i)	7	1	
	(ii)	$\frac{8-2x}{3}$ oe final answer	2*	**B1** for $3x = 8 - 2y$ or $3y = 8 - 2x$ or $2x = 8 - 3y$ or $2y = 8 - 3x$ or $1.5x = 4 - y$ or $1.5y = 4 - x$ or $\frac{8-2x}{3}$ oe seen or $\frac{8-2y}{3}$ oe seen

18	**(a)**	$-\frac{3}{5}$, or -0.6	1

19	**(a)**	–27 –8 –1 0 1 8 27	1	
	(b)	7 correct plots and smooth curve	2	**B1** for 5 plots
	(c) (i)	– 2.4 to – 2.6	1	
	(ii)	4 to 6	1	
	(iii)	$t = u^3$	1	
	(iv)	10 to 13	2	**M1** for a tangent at $x = 2$
	(d) (i)	Correct line	2	**B1** for correct intercept (0, 3) or gradient 5
	(ii)	(–1.95 to –1.7) (– 0.8 to –0.5) (2.4 to 2.6)	2	**B1** for one correct